On the zero sets of bounded holomorphic functions in the bidisc

by

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Abstract

In this work we prove in a constructive way a theorem of Rudin which says that if E is an analytic subset of the bidisc D^2 (with multiplicities) which does not intersect a neighbourhood of the distinguished boundary, then E is the zero set (with multiplicities) of a bounded holomorphic function. This approach allows us to generalize this theorem and also some results obtained by P.S. Chee.

I. Introduction and statement of the results

Let $H^\infty(D^n)$ be the algebra of bounded holomorphic functions in the polydisc. Very few results are known on the analytic sets wich are zero sets of functions in $H^\infty(D^n)$. Some non trivial examples of such sets were given by W. Rudin in 1967 [Ru1] and P.S. Chee in 1976 [Che]. Rudin showed that if E is an analytic set in the polydisc $D^n = \{z = (z_1, \dots, z_n) \,/\, |z_i| < 1\,,\ 1 \le i \le n\}$ such that the intersection of E with a neighbourhood of $\mathbb{T}^n$, where

$$\mathbb{T}^n = \{z = (z_1, \dots, z_n) \,/\, |z_i| = 1\,,\ 1 \le i \le n\},$$

is empty then E is the zero set of a bounded holomorphic function in D^n (counting multiplicity).

[1] This work was done during the visit of the first-named author to the Centre de Recerca Matemàtica at Barcelona. He wants to thank this institution for its support.

[2] Partially supported by DGICYT grant PB92-0804-C02-01 and grant HF94-006 of the "Programa de acciones integradas hispano-francesas"

A few years later in 1974, S. Zarantonello [Za] proved that if E is an analytic set in D^n such that there exist an $r \in (0,1)$ and a continuous function $\eta : [r,1) \to [r,1)$ such that, for all $z = (z_1, \ldots, z_n)$ belonging to $E \cap \{z \in D^n \,/\, |z_i| \geq r\,,\ 1 \leq i \leq n\}$ we have

$$|z_n| \leq \eta \left(\frac{|z_1| + \cdots + |z_{n-1}|}{n-1} \right),$$

then E is the zero set of a function F of the usual Nevanlinna class of D^n (i.e. $\sup_{0<r<1} \int_{\mathbb{T}^n} \log^+ |F(rz_1, \ldots, rz_n)| d\sigma < +\infty$). A question posed by S. Zarantonello in his paper was whether under the same hypothesis a bounded function F can be taken. Chee in 1976 [Che], gave an affirmative answer to that question.

The same problem in the unit ball $\mathbb{B}$ of $\mathbb{C}^2$ was considered by B. Berndtsson in 1980 [Be] and he proved that if E is an analytic subset of $\mathbb{B}$ of finite area (with multiplicity) then it can be defined by a bounded holomorphic function. In his proof he used the connection between the zero sets of holomorphic functions and the equation

$$i\partial\overline{\partial}u = \theta\,, \tag{1}$$

where θ is a positive closed $(1,1)$-current, found by P. Lelong [Le]. P. Lelong proved that to each analytic set with multiplicities, i.e. to each divisor, E there is an associated $(1,1)$ current

$$\theta_E = i \sum \theta_{ij}\, d\xi_i \wedge d\overline{\xi}_j$$

which is positive and closed (i.e. $d\,\theta_E = 0$), and showed that any solution u of (1), with $\theta = \theta_E$, can be written as $u = \log|f|$, where f vanishes exactly in E with the given multiplicities. Thus if we can find a solution u of (1) which is bounded from above, we have a bounded homomorphic function which defines the divisor E. We will denote by Supp E the support of the associated $(1,1)$ current θ_E.

Here, we will use this method to prove the following:

Theorem 1 *Let E be an analytic subset of D^2 with multiplicities, i.e. a divisor in D^2. Suppose that there exist two continuous functions $\eta_1, \eta_2 : [0,1) \to [0,1)$, $\lim_{t \to 1} \eta_i(t) = 1$ such that*

$$\text{Supp } E \cap \Big\{(z_1, z_2) \in D^2 \,/\, |z_1| = \eta_1(t),\ |z_2| = \eta_2(t),\ t \in [0,1)\Big\} = \emptyset$$

then E is the divisor associated to a bounded holomorphic function in D^2.

Remarks

1. Observe that any Rudin variety satisfies the hypothesis of theorem 1 and also the varieties considered by Zarantonello and Chee.

2. One can give an analogous result for the polydisc (in the case $n \geq 3$), but the computation involved is more tedious.

As the following example shows, the condition of finite area of E is not sufficient for the existence of a bounded holomorphic function which vanishes on E.

Let a_i be a sequence in D such that

$$\sum_{i=1}^{\infty}(1-|a_i|)^{3/2} < +\infty \quad \text{but} \quad \sum_{i=1}^{\infty}(1-|a_i|) = +\infty\,.$$

Consider $E = \cup_{i=1}^{\infty} E_i$ with $E_i = \{(z_1, z_2) \in D^2 \text{ s.t. } \quad z_1 + z_2 = 2a_i\}$. Then the area of E is comparable to

$$\sum_{i=1}^{\infty}(1-|a_i|)^{3/2}\,.$$

If there were a function $f \in H^\infty(D^2)$ such that f vanished on E, then $g : D \to D$, $g(z) = f(z, z)$ is bounded and its zeros are $\{a_i\}$ which do not satisfy the Blaschke condition. This example was previously considered in [Ch1]. In fact in [Ch2] (see also [Ch3]) it is proved that the finite area condition for a divisor in D^2 is suficient to assure the existence of a function belonging to the Nevanlinna class and defining the given divisor, and, in this particular example, which consists of a union of hyperplanes, the finite area condition is also necessary to assure the existence of a function in the Nevanlinna class with zeros the hyperplanes.

Nevertheless there are zero sets E of infinite area which satisfy the hypothesis of theorem 1, they are even Rudin varieties (i.e. they are far from $\mathbb{T}^2$). Consider for instance the analytic disc defined by

$$f : D \to D^2\,; \quad f(z) = (z, \frac{1}{2}g(z))$$

where g is any inner function of the disc different from a finite Blaschke product. As $\|f_2(z)\| \leq \frac{1}{2}$ the analytic disc is far from the distinguished boundary. The area of the variety is comparable to the sum of the areas of the projections on the axis (counting multiplicity). But it can be proved, see for instance theorem 6.6 of [Ga], that given any inner function g different from a finite Blaschke product, then, there exists a set $L \subset D$ of logarithmic capacity 0, such that for all $z \in D \setminus L$, card $(g^{-1}(z)) = \infty$. So the projection of the analytic disc in the z_2-axis is a disc centered in zero and of radius $\frac{1}{2}$ with infinite multiplicity, (possibly the whole disc minus L). Therefore it has infinite area.

Now we can observe that W. Rudin's result can be stated as follows: If X is a divisor in D^2 that in a neighbourhood of $\mathbb{T}^2$ is equal to the trivial divisor associated to the constant function 1 then it is defined by a bounded function. H. Alexander asked us whether the same result is true if we substitute the function 1 by any bounded holomorphic function.

In this direction we can prove the following:

Proposition 2 *Let X be a divisor in D^n. If there exists a divisor Y associated to a function $h \in H^\infty(D^n)$ and a neighbourhood of $\mathbb{T}^n$, $\vartheta(\mathbb{T}^n)$ such that*

$$X \cap \vartheta(\mathbb{T}^n) = Y \cap \vartheta(\mathbb{T}^n)$$

then X is the divisor associated to some bounded holomorphic function in D^n.

In the bidisc we can prove also

Theorem 3 *Let X be a divisor in D^2. Suppose that there exists a function $h \in H^\infty(D^2)$ and an $r < 1$ such that, if Y is the divisor associated to h then*

$$X_{|\Delta_r} = Y_{|\Delta_r},$$

where $\Delta_r = \{(z_1, z_2) \in D^2 / \ r < |z_1| = |z_2| < 1\}$. Then X is contained in a divisor associated to a bounded holomorphic function.

Remarks

1. The same result remains true if we substitue $\Delta = \{(z_1, z_2) \in D^2 / |z_1| = |z_2|\}$ by $\{(z_1, z_2) \in D^2 / |z_1| + (\alpha - 1)|z_2| = \alpha\}$, $\alpha \in [0, 1)$.

2. In theorem 3 we cannot assure that the divisor is equal to one defined by a bounded holomorphic function as the next example, which has been previously considered by E. Amar, shows. Let $f \in L^2(D) \cap H(D)$, with zeros a_n that do not satisfy the Blaschke condition, i.e.

$$\sum_{i=1}^{\infty} (1 - |a_i|) = \infty.$$

Let $g(z_1, z_2) = f\left(\dfrac{z_1 + z_2^2}{2}\right)$. Consider $V = Z(g)$, where $Z(g)$ denotes the zero set of g. Suppose that there is a bounded holomorphic function h such that $V = Z(h)$. Then $H(\xi) = h(\xi^2, \xi)$ is an holomorphic bounded function in the disc, but its zeros do not satisfy the Blaschke condition ($H(\xi) = 0 \Longleftrightarrow \xi^2 = a_i$), therefore such an h does not exist. Now consider

$$k(z_1, z_2) = f\left(\frac{z_1 + z_2^2}{2}\right)(z_2^2 - z_1)^2.$$

As $|f(z)| \leq 4\|f\|_{L^2(D)}(1 - |z|^2)^{-1}$, see for instance theorem 7.2.5 of [Ru2]. Then

$$|k(z_1, z_2)| \leq 4\|f\|_{L^2(D)}|z_2^2 - z_1|^2 \left(1 - \left|\frac{z_1 + z_2^2}{2}\right|^2\right)^{-1} \leq 16\|f\|_{L^2(D)}.$$

So k is a bounded holomorphic function and $Z(k) = V \cup 2\{z_2^2 = z_1\}$. So V is contained in the zero set of k. Note that it intersects Δ in the same set as $Z(k)$. In fact it coincides with $Z(k)$ outside $\{z_2^2 = z_1\}$.

II. Proof of theorem 1

Let E be a divisor in D^2 which satisfies the hypothesis of theorem 1 and let $\theta = i \sum_{i,j=1}^{2} \theta_{ij}\, d\xi_i \wedge d\overline{\xi}_j$ be the $(1,1)$-closed positive current associated. We want to solve the equation (1) with an upper bound for the solution.

This bound will be directly related with the following elementary lemma (which was also used in [Ru1] and [Za]):

Lemma 1 *Under the hypothesis of theorem 1, there exist two constants $M > 0, N > 0$ such that for all $t \in [0,1)$, there is a neighbourhood ϑ_t of $\mathbb{T}_{\eta_1(t)} \times \mathbb{T}_{\eta_2(t)} = \{(z_1, z_2) \in D^2 \,/\, |z_1| = \eta_1(t),\ |z_2| = \eta_2(t)\}$ with*

$$\int_{|\xi_1|<\eta_1(t)} \theta_{11}(\xi_1, z_2) = M, \quad \int_{|\xi_2|<\eta_2(t)} \theta_{22}(z_1, \xi_2) = N, \qquad \forall\ (z_1, z_2) \in \vartheta_t.$$

Proof: As E is a divisor in D^2, there is an holomorphic function h, such that defines E, i.e. $i\partial\overline{\partial} \log |h| = \theta$. Let $n_1(z_1, t)$ be the number of zeros of h in the disc $\{\xi_1 = z_1,\ |\xi_2| < \eta_2(t)\}$. By the argument principle

$$n_1(z_1, t) = \frac{1}{2\pi i} \int_{|\xi_2|=\eta_2(t)} \frac{h_{z_2}(z_1, \xi_2)}{h(z_1, \xi_2)}\, d\xi_2 = \int_{|\xi_2|<\eta_2(t)} \theta_{22}(z_1, \xi_2)\,.$$

Similarly if $n_2(x_2, t)$, $z_2 \in D$, is the number of zeros of h in the disc $\{|\xi_1| < \eta_1(t)\,,\ \xi_2 = z_2\}$ then

$$n_2(z_2, t) = \frac{1}{2\pi i} \int_{|\xi_1|=\eta_1(t)} \frac{h_{z_1}(\xi_1, z_2)}{h(\xi_1, z_2)}\, d\xi_1 = \int_{|\xi_1|<\eta_1(t)} \theta_{11}(\xi_1, z_2)\,.$$

We choose ϑ_t such that the support of θ does not intersect it. As long as $(z_1, z_2) \in \vartheta_t$, both functions $n_1(z_1, t)$ and $n_2(z_2, t)$ are continuous in z_i and in t because $h(\xi_1, \xi_2) \neq 0$ when $(\xi_1, \xi_2) \in \vartheta_t$ for any $t \in [0,1)$. As they are integer valued functions, $n_1(z_1, t)$ and $n_2(z_2, t)$ are constant for $(z_1, z_2) \in \mathbb{T}_{\eta_1(t)} \times \mathbb{T}_{\eta_2(t)}$, $t \in [0,1)$.

That means that

$$n_1(z_1, t) = N, \qquad n_2(z_2, t) = M,$$

which was the desired result. ∎

Later on, in order to assure the convergence in a regularization process we will need that the current satisfies the Blaschke condition. The next lemma takes care of it.

Lemma 2 *If we have a divisor E with associated $(1,1)$ current θ such that it satisfies the conclusion of lemma 1, i.e.*

$$\int_{|\xi_1|<\eta_1(t)} \theta_{11}(\xi_1, z_2) = M, \quad \int_{|\xi_2|<\eta_2(t)} \theta_{22}(z_1, \xi_2) = N, \qquad \forall\ (z_1, z_2) \in \vartheta_t$$

and the support of θ does not intersect ϑ_t, then the divisor E satisfies the Blaschke condition, i.e. if $\delta_{\partial D^2}$ is the distance to the boundary of D^2:

$$\int_{D^2} \delta_{\partial D^2} |\theta| < \infty.$$

Proof: Let f be an holomorphic function such that it defines the divisor E, i.e. $\theta = i\partial\overline{\partial} \log |f|$, by [Ch1], it suffices to prove that

$$\sup_{r<1} \int_{\mathbb{T}_r \times \mathbb{T}_r} \log |f|\, d\sigma < +\infty.$$

We use the Jensen formula in the following way: if u is an holomorphic function in D, for any $0 < r_0 < r < 1$,

$$\int_{\mathbb{T}_r} \log |u|\, d\sigma - \int_{\mathbb{T}_{r_0}} \log |u|\, d\sigma = \int_{r_0}^{r} \frac{dt}{t} \left(\int_{D_t} \Delta(\log |u|) \right). \tag{2}$$

Now, we fix t_0, and take t big enough, such that $0 < t_0 < t < 1$ and $\eta_j(t) > \eta_j(t_0)$, $j = 1, 2$.

We make a partition of the parameter interval $t_0 < \cdots < t_n = t$ such that for any $0 \leq i < n$ the set

$$S_i = \{(z_1, z_2);\ \min(\eta_j(t_i), \eta_j(t_{i+1})) \leq |z_j| \leq \max(\eta_j(t_i), \eta_j(t_{i+1})),\ j = 1, 2\}$$

does not intersect the suport of θ. We fix $0 \leq i < n$ and we consider

$$\begin{aligned} \int_{\mathbb{T}_{\eta_1(t_{i+1})} \times \mathbb{T}_{\eta_2(t_{i+1})}} & \log |f|\, d\sigma - \int_{\mathbb{T}_{\eta_1(t_i)} \times \mathbb{T}_{\eta_2(t_i)}} \log |f|\, d\sigma = \\ = \quad & \int_{\mathbb{T}_{\eta_1(t_{i+1})} \times \mathbb{T}_{\eta_2(t_{i+1})}} \log |f|\, d\sigma - \int_{\mathbb{T}_{\eta_1(t_i)} \times \mathbb{T}_{\eta_2(t_{i+1})}} \log |f|\, d\sigma + \\ & + \int_{\mathbb{T}_{\eta_1(t_i)} \times \mathbb{T}_{\eta_2(t_{i+1})}} \log |f|\, d\sigma - \int_{\mathbb{T}_{\eta_1(t_i)} \times \mathbb{T}_{\eta_2(t_i)}} \log |f|\, d\sigma. \end{aligned} \tag{3}$$

Define A to be the difference of the first two integrals of the right hand side member of equality (3) and B to be the difference of the last two integrals of (3). Applying Jensen's formula (2) to B, we get

$$B = \int_{\xi_1 \in \mathbb{T}_{\eta_1(t_i)}} \left\{ \int_{\eta_2(t_i)}^{\eta_2(t_{i+1})} \frac{ds}{s} \left(\int_{\xi_2 \in D_s} \theta_{22}(\xi_1, \xi_2) \right) \right\}.$$

As the support of θ does not intersect S_i then for any $|\xi_1| = |\eta_1(t_i)|$ and any $s \in [\min(\eta_2(t_i), \eta_2(t_{i+1})), \max(\eta_2(t_i), \eta_2(t_{i+1}))]$ we have

$$\int_{\xi_2 \in D_s} \theta_{22}(\xi_1, \xi_2) = \int_{\xi_2 \in D_{\eta_2(t_i)}} \theta_{22}(\xi_1, \xi_2).$$

Thus, because of the hypothesis of the lemma we get

$$B = N \int_{\eta_2(t_i)}^{\eta_2(t_{i+1})} \frac{ds}{s}. \tag{4}$$

Now we estimate A applying again Jensen's formula

$$A = \int_{\xi_2 \in \mathbb{T}_{\eta_2(t_{i+1})}} \left\{ \int_{\eta_1(t_i)}^{\eta_1(t_{i+1})} \frac{ds}{s} \left(\int_{\xi_1 \in D_s} \theta_{11}(\xi_1, \xi_2) \right) \right\}.$$

Just like before we get

$$A = M \int_{\eta_1(t_i)}^{\eta_1(t_{i+1})} \frac{ds}{s}. \tag{5}$$

We consider now

$$\begin{aligned} &\int_{\mathbb{T}_{\eta_1(t)} \times \mathbb{T}_{\eta_2(t)}} \log|f|\, d\sigma - \int_{\mathbb{T}_{\eta_1(t_0)} \times \mathbb{T}_{\eta_2(t_0)}} \log|f|\, d\sigma = \\ &= \sum_{i=0}^{n-1} \int_{\mathbb{T}_{\eta_1(t_{i+1})} \times \mathbb{T}_{\eta_2(t_{i+1})}} \log|f|\, d\sigma - \int_{\mathbb{T}_{\eta_1(t_i)} \times \mathbb{T}_{\eta_2(t_i)}} \log|f|\, d\sigma. \end{aligned}$$

Now in each term of the sum we can compute with (4) and (5) and get

$$\begin{aligned} &\int_{\mathbb{T}_{\eta_1(t)} \times \mathbb{T}_{\eta_2(t)}} \log|f|\, d\sigma - \int_{\mathbb{T}_{\eta_1(t_0)} \times \mathbb{T}_{\eta_2(t_0)}} \log|f|\, d\sigma = \\ &M \int_{\eta_1(t_0)}^{\eta_1(t)} \frac{ds}{s} + N \int_{\eta_2(t_0)}^{\eta_2(t)} \frac{ds}{s} \le M \log \frac{1}{\eta_1(t_0)} + N \log \frac{1}{\eta_2(t_0)}. \end{aligned} \tag{6}$$

Define $\eta(t)$ to be $\eta(t) = \min(\eta_1(t), \eta_2(t))$. Then, by the subharmonicity and using (6) we obtain

$$\begin{aligned} &\int_{\mathbb{T}_{\eta(t)} \times \mathbb{T}_{\eta(t)}} \log|f| \le \int_{\mathbb{T}_{\eta_1(t)} \times \mathbb{T}_{\eta_2(t)}} \log|f| \le \\ &\le C(\eta_1(t_0), \eta_2(t_0)) \left(M + N + \int_{\mathbb{T}_{\eta_1(t_0)} \times \mathbb{T}_{\eta_2(t_0)}} \log|f| \right) \le C < +\infty. \blacksquare \end{aligned}$$

Using lemma 1 and 2, theorem 1 is a special case of the following theorem.

Theorem 1'. *Let* $\theta = i \sum_{i,j=1}^{2} \theta_{ij}\, d\xi_i \wedge d\overline{\xi}_j$ *be a closed positive current in* D^2. *Suppose that there are two sequences* (r_n^1) *and* (r_n^2), $r_n^i \in [0,1)$, $\lim_{n\to\infty} r_n^i = 1$, $i = 1,2$ *such that*

(a) *For all* n, $\operatorname{Supp}\theta_{12} \cap (\mathbb{T}_{r_n^1} \times \mathbb{T}_{r_n^2}) = \emptyset$.

(b) *There is an* $M > 0$ *such that for all* n *there is a neighbourhood* ϑ_n *of* $\mathbb{T}_{r_n^1} \times \mathbb{T}_{r_n^2}$ *such that*

$$\sup_{(z_1,z_2)\in\vartheta_n} \left\{ \int_{|\xi_1|<r_n^1} \theta_{11}(\xi_1, z_2) + \int_{|\xi_2|<r_n^2} \theta_{22}(z_1,\xi_2) \right\} \leq M.$$

(c)

$$A = \int_{D^2} \delta_{\partial D^2} |\theta| < \infty.$$

Then there exists a negative solution u *to the equation* $i\partial\overline{\partial}u = \theta$.

To prove this statement, we will first construct an explicit solution of the equation (1) whose boundary values on $\mathbb{T}^2$ are well adapted to (a), (b) and (c). Theorem 1' will then follow using an appropriate regularization process.

1. An explicit expression for the boundary values of a solution of (1)

In this section we will work with a closed positive $(1,1)$ current with coefficients in $C^\infty(\overline{D^2})$.

First, using the method developed by M. Anderson in [And], we write down the solution of (1) with minimal $L^2(\mathbb{T}^2)$ norm. Then we will modify this solution by adding some pluriharmonic functions to obtain a "good" expression for the boundary values:

Lemma 3 *Let* θ *be a closed* $(1,1)$ *real form with coefficients in* $C^\infty(\overline{D^2})$. *Then the function* $\widetilde{M}(\theta)$ *defined on* $\mathbb{T}^2$ *by*

$$\begin{aligned}\widetilde{M}(\theta)(z_1,z_2) &= -\frac{1}{2\pi^2}\operatorname{Re}\left\{ i\int_{\xi\in\Delta} d(\log(1-\xi_1\overline{z_1})\log(1-\overline{\xi_2}z_2)\theta(\xi_1,\xi_2)) \right\} + \\ &\quad + \frac{1}{\pi}\left\{ \int_{|\xi_1|\leq 1} \log|1-\xi_1\overline{z_1}|\, i\theta_{11}(\xi_1,z_2)\, d\xi_1\wedge d\overline{\xi_1} + \right. \\ &\quad \left. + \int_{|\xi_2|\leq 1} \log|1-\overline{\xi_2}z_2|\, i\theta_{22}(z_1,\xi_2)\, d\xi_2\wedge d\overline{\xi_2} \right\},\end{aligned}$$

is the boundary values on $\mathbb{T}^2$ *of a solution (which will still be denoted by* $\widetilde{M}(\theta)$*) of the equation* $i\partial\overline{\partial}u = \theta$ *that belongs to* $C^\infty(\overline{D^2})$.

Proof: In [And], M. Andersson finds the solution u of (1) with minimal $L^2(d\lambda^\alpha)$-norm, where $d\lambda^\alpha = (1-|\lambda_1|)^{\alpha_1} d\lambda_1 (1-|\lambda_2|)^{\alpha_2} d\lambda_2$ and $d\lambda$ is the Lebesgue measure in the disc. In fact, the integral kernel which solves (1) with minimal $L^2(\mathbb{T}^2)$ norm can be obtained formally from the Andersson kernel letting $\alpha_1 \to -1$ and $\alpha_2 \to -1$. For the sake of completeness, let us recall this construction.

Let S denote the Szegö projection from $L^2(\mathbb{T}^2)$ to $H^2(\mathbb{T}^2)$. Let us define $\overline{S}$ by $\overline{S}\varphi = \overline{S\overline{\varphi}}$ and S^0 by $S^0\varphi = (S\varphi)(0,0)$. This last is correctly defined, as any function in $H^2(\mathbb{T}^2)$ can be extended holomorphically to D^2 via its Poisson integral. If we consider $\Pi = S + \overline{S} - S^0$ then $\Pi\varphi$ is pluriharmonic in fact it is the orthogonal projection from $L^2(\mathbb{T}^2)$ to $L^2(\mathbb{T}^2) \cap \{u\,;\ u \text{ is pluriharmonic in } D^2\}$. Let u be a solution of (1). Since Πu is pluriharmonic $u - \Pi u$ depends only on $i\partial\overline{\partial}u = \theta$, so we can define an operator solution which gives us the solution of (1) with minimal $L^2(\mathbb{T}^2)$ norm:

$$M\theta = u - \Pi u\,. \tag{7}$$

Now, we want to find an explicit integral formula for $M(\theta)$. In order to do this, one must decompose M as a sum of operators which operate coordinatewise and are of adequate bidegree. We introduce now the needed operators.

$$K\overline{\partial}u = u - Su\,. \tag{8}$$

K is the solution operator which solves the $\overline{\partial}$-equation with minimal $L^2(\mathbb{T}^2)$ norm. We need also

$$T\overline{\partial}u = \overline{S}u - S^0 u\,. \tag{9}$$

$\overline{T}$ and $\overline{K}$ are the conjugate operators defined just like $\overline{S}$.

Now in terms of these operators, the solution operator M can be written as

$$M(i\partial\overline{\partial}u) = K\overline{\partial}u - T\overline{\partial}u = \overline{K}\partial u - \overline{T}\partial u$$

because

$$M = I - \Pi = I - (S + \overline{S} - S^0) = I - S - (\overline{S} - S^0)$$

and M is real (i.e. $\overline{M(\overline{\Theta})} = M(\Theta)$).

The explicit formulae for I, K, T, S, S^0 and M are well-known in one variable, but as Andersson shows, one can find the explicit expression of the operators in $D^2 = D_1 \times D_2$ if we know the expression of the operators in each variable. For instance

$$K(\overline{\partial}u) = u - Su = (I_1I_2 - S_1S_2)u = I_1(I_2 - S_2)u + (I_1 - S_1)S_2u = I_1K_1(\overline{\partial_2}u) + K_1S_2(\overline{\partial_1}u)\,.$$

Similarly

$$M = S_1^0 M_2 + M_1 S_2^0 + \overline{T_1}K_2 + \overline{K_1}T_2 - \overline{T_1}T_2 + K_1\overline{K_2}\,. \tag{10}$$

This last expression of M has the advantage that each term of the sum acts on $\partial\overline{\partial}u$ because of bidegree reasons. For instance $K_1\overline{K_2}$ acts on $\theta_{21} = \partial_2\overline{\partial_1}u$. To prove (10) one substitutes the operators K, T, M in (10) by formulae (7), (8) and (9) and gets that

$$M = I_1I_2 - S_1S_2 - \overline{S_1S_2} + S_1^0 S_2^0$$

which is exactly the definition. Expression (10) is not symmetric but since M is a real operator $M = \frac{1}{2}(M + \overline{M})$, therefore

$$\begin{aligned} M &= S_1^0 M_2 + M_1 S_2^0 + \frac{1}{2}(\overline{T_1}K_2 + \overline{K_1}T_2 - \overline{T_1}T_2 + \overline{K_1}K_2) + \\ &\quad + \frac{1}{2}(T_1\overline{K_2} + K_1\overline{T_2} - T_1\overline{T_2} + K_1\overline{K_2}) \end{aligned} \tag{11}$$

and this is the expression that we will compute explicitely. We write down now the integral expression for each operator in one variable that appears in (11).

If we have a smooth function u in $\overline{D}$ then the Szegö projection is

$$S(u)(z) = \int_{\mathbb{T}} \frac{1}{2\pi i} \frac{1}{\xi - z} u(\xi)\, d\xi$$

and consequently

$$S^0(u)(z) = \int_{\mathbb{T}} \frac{1}{2\pi i} u(\xi)\overline{\xi}\, d\xi\,. \tag{12}$$

Let us suppose that $\theta = i\theta_{11} d\xi \wedge d\overline{\xi}$, then the Poisson-Jensen formula states that

$$M(\theta)(z) = \int_{|\xi|<1} \frac{1}{2\pi} \log\left|\frac{\xi - z}{1 - \overline{\xi} z}\right|^2 \theta(\xi)\,. \tag{13}$$

If $\overline{\partial} u = w$ is a smooth $(0,1)$-form, then the Cauchy-Green formula states that

$$K(w)(z) = \int_{|\xi|<1} \frac{i}{2\pi} \frac{d\xi}{\xi - z} \wedge w(\xi)\,. \tag{14}$$

Similarly

$$T(w)(z) = \int_{|\xi|<1} \frac{i}{2\pi} \frac{\overline{z} d\xi}{(1 - \xi\overline{z})} \wedge w(\xi)\,. \tag{15}$$

The solution $M(\theta)$ in D^2 can be now written applying (12), (13), (14) and (15) in (11) as:

$$M(\theta)(z) = \int_{\xi \in D^2} m_0(\xi, z) \wedge \theta(\xi) + \int_{\xi \in \mathbb{T}\times D} m_1(\xi, z) \wedge \theta(\xi) + \int_{\xi \in D\times\mathbb{T}} m_2(\xi, z) \wedge \theta(\xi)$$

where

$$m_1(\xi, z) = \frac{1}{4\pi^2 i}\overline{\xi_1} \log\left|\frac{\xi_2 - z_2}{1 - \overline{\xi_2} z_2}\right|^2 d\xi_1$$

$$m_2(\xi, z) = \frac{1}{4\pi^2 i}\overline{\xi_2} \log\left|\frac{\xi_1 - z_1}{1 - \overline{\xi_1} z_1}\right|^2 d\xi_2$$

and

$$\begin{aligned} m_0(\xi,z) \;\; = \;\; & \frac{1}{8\pi^2 i}\left(\frac{z_1}{(1-\overline{\xi_1}z_1)(\xi_2-z_2)}+\frac{\overline{z_2}}{(\overline{\xi_1}-\overline{z_1})(1-\xi_2\overline{z_2})}-\right. \\ & \left. -\frac{z_1\overline{z_2}}{(1-\overline{\xi_1}z_1)(1-\xi_2\overline{z_2})}+\frac{1}{(\overline{\xi_1}-\overline{z_1})(\xi_2-z_2)}\right)\,d\overline{\xi_1}\wedge d\xi_2 \\ & +\frac{1}{8\pi^2 i}\left(\frac{\overline{z_1}}{(1-\xi_1\overline{z_1})(\overline{\xi_2}-\overline{z_2})}+\frac{z_2}{(\xi_1-z_1)(1-\overline{\xi_2}z_2)}-\right. \\ & \left. -\frac{\overline{z_1}z_2}{(1-\xi_1\overline{z_1})(1-\overline{\xi_2}z_2)}+\frac{1}{(\xi_1-z_1)(\overline{\xi_2}-\overline{z_2})}\right)\,d\overline{\xi_2}\wedge d\xi_1\,. \end{aligned}$$

Then $M(\theta)\in C^\infty(\overline{D^2})$ and if we consider only the values at the distinguished boundary $|z_1|=|z_2|=1$ the expression becomes simpler:

$$M(\theta)(z_1,z_2)=\frac{i}{4\pi^2}\int_{\xi\in D^2}\left(\frac{d\overline{\xi_1}\wedge d\xi_2}{(\overline{z_1}-\overline{\xi_1})(z_2-\xi_2)}+\frac{d\overline{\xi_2}\wedge d\xi_1}{(z_1-\xi_1)(\overline{z_2}-\overline{\xi_2})}\right)\wedge\theta\,,\quad z\in\mathbb{T}^2\,,$$

or equivalently

$$M(\theta)(z_1,z_2)=-\frac{1}{2\pi^2}\,\mathrm{Re}\,\left\{i\int_{\xi\in D^2}\frac{d\xi_1}{z_1-\xi_1}\wedge\frac{d\overline{\xi_2}}{\overline{z_2}-\overline{\xi_2}}\wedge\theta\right\}\qquad z\in\mathbb{T}^2\,.$$

We now modify this solution of (1) by adding some pluriharmonic functions in $C^\infty(\overline{D^2})$, so we will still have a smooth solution of (1). Consider

$$\begin{aligned} \int_{\xi\in D^2}\frac{d\xi_1}{z_1-\xi_1}\wedge\frac{d\overline{\xi_2}}{\overline{z_2}-\overline{\xi_2}}\wedge\theta &= \int_{|\xi_1|<|\xi_2|}\frac{\overline{z_1}}{1-\xi_1\overline{z_1}}\,d\xi_1\wedge\frac{d\overline{\xi_2}}{\overline{z_2}-\overline{\xi_2}}\wedge\theta+ \\ &+\int_{|\xi_2|<|\xi_1|}\frac{d\xi_1}{z_1-\xi_1}\wedge\frac{z_2 d\overline{\xi_2}}{1-\overline{\xi_2}z_2}\wedge\theta\,. \end{aligned}\tag{16}$$

We look now at the third integral of (16)

$$\begin{aligned} \int_{|\xi_2|<|\xi_1|}\theta\wedge\frac{d\xi_1}{z_1-\xi_1}\wedge\frac{z_2 d\overline{\xi_2}}{1-\overline{\xi_2}z_2} &= \int_{|\xi_2|<|\xi_1|}\frac{(1-|\xi_1|^2)d\xi_1}{(z_1-\xi_1)(1-\overline{\xi_1}z_1)}\wedge\frac{z_2 d\overline{\xi_2}}{1-\overline{\xi_2}z_2}\wedge\theta- \\ &-\int_{|\xi_2|<|\xi_1|}\theta\wedge\frac{\overline{\xi_1}d\xi_1}{(1-\overline{\xi_1}z_1)}\wedge\frac{z_2 d\overline{\xi_2}}{1-\overline{\xi_2}z_2}\,. \end{aligned}\tag{17}$$

The second integral in the right hand side of (17) are the boundary values in $\mathbb{T}^2$ of the holomorphic function in $A^\infty(D^2)$

$$f_1(z_1,z_2)=\int_{|\xi_2|<|\xi_1|}\theta\wedge\frac{\overline{\xi_1}d\xi_1}{(1-\overline{\xi_1}z_1)}\wedge\frac{z_2 d\overline{\xi_2}}{1-\overline{\xi_2}z_2}\,.$$

In the first integral in the right hand side of (17) consider the values of z_1, z_2 extended to the interior of D^2, as θ is $C^\infty(\overline{D^2})$, then the integral as a function of z_1, z_2 is C^∞ up to the boundary. So we consider now $(z_1, z_2) \in D^2$. We denote by $B_\varepsilon = \{(\xi_1, \xi_2) \in D^2\,;\ |\xi_1 - z_1| < \varepsilon\}$, taking ε such that $|z_1| + \varepsilon < 1$. We have that

$$\int_{|\xi_2|<|\xi_1|} \theta \wedge \frac{(1-|\xi_1|^2)d\xi_1}{(z_1-\xi_1)(1-\overline{\xi_1}z_1)} \wedge \frac{z_2 d\overline{\xi_2}}{1-\overline{\xi_2}z_2} =$$

$$= \int_{(|\xi_2|<|\xi_1|)\setminus B_\varepsilon} \theta \wedge \frac{(1-|\xi_1|^2)d\xi_1}{(z_1-\xi_1)(1-\overline{\xi_1}z_1)} \wedge \frac{z_2 d\overline{\xi_2}}{1-\overline{\xi_2}z_2} +$$

$$+ \int_{\substack{B_\varepsilon \cap \\ \{|\xi_2|<|\xi_1|\}}} \theta \wedge \frac{(1-|\xi_1|^2)d\xi_1}{(z_1-\xi_1)(1-\overline{\xi_1}z_1)} \wedge \frac{z_2 d\overline{\xi_2}}{1-\overline{\xi_2}z_2}\,.$$

Recall that we denote by Δ the set $\{(\xi_1, \xi_2) \in D^2;\ |\xi_1| = |\xi_2|\}$. Also, we will denote by $\int_A^* w = \int_{A\setminus B_\varepsilon} w$. With this notation and applying Stokes' formula,

$$\int^*_{|\xi_2|<|\xi_1|} \theta \wedge \frac{(1-|\xi_1|^2)d\xi_1}{(z_1-\xi_1)(1-\overline{\xi_1}z_1)} \wedge \frac{z_2 d\overline{\xi_2}}{(1-\overline{\xi_2}z_2)} =$$

$$= \int^*_{|\xi_2|<|\xi_1|} d\left(\frac{(1-|\xi_1|^2)d\xi_1}{(z_1-\xi_1)(1-\overline{\xi_1}z_1)} \log(1-\overline{\xi_2}z_2) \wedge \theta\right) -$$

$$- \int^*_{|\xi_2|<|\xi_1|} \frac{d\overline{\xi_1} \wedge d\xi_1}{(1-\overline{\xi_1}z_1)^2} \log(1-\overline{\xi_2}z_2) \wedge \theta =$$

$$= \int^*_{\xi\in\Delta} \frac{(1-|\xi_1|^2)d\xi_1}{(z_1-\xi_1)(1-\overline{\xi_1}z_1)} \log(1-\overline{\xi_2}z_2) \wedge \theta -$$

$$- \int_{\substack{|z_1-\xi_1|=\varepsilon \\ |\xi_2|<|\xi_1|}} \frac{(1-|\xi_1|^2)d\xi_1}{(z_1-\xi_1)(1-\overline{\xi_1}z_1)} \log(1-\overline{\xi_2}z_2) \wedge \theta -$$

$$- \int^*_{|\xi_2|<|\xi_1|} \frac{d\overline{\xi_1} \wedge d\xi_1}{(1-\overline{\xi_1}z_1)^2} \log(1-\overline{\xi_2}z_2) \wedge \theta\,.$$

If we let $\varepsilon \to 0$ the star dissapears in all integrals except from the second term of the last member:

$$\lim_{\varepsilon\to 0}\left[\int_{|z_1-\xi_1|=\varepsilon} \frac{d\xi_1}{\xi_1 - z_1}\left\{\int_{|\xi_2|<|\xi_1|} \frac{(1-|\xi_1|^2)}{1-\overline{\xi_1}z_1} \log(1-\overline{\xi_2}z_2) \wedge \theta\right\}\right] =$$

$$= 2\pi i \int_{|\xi_2|<|z_1|} \log(1-\overline{\xi_2}z_2) i\theta_{22}(z_1, \xi_2) d\xi_2 \wedge d\overline{\xi_2}$$

So we have for any $(z_1, z_2) \in D^2$

$$
\begin{aligned}
&\int_{|\xi_2|<|\xi_1|} \theta \wedge \frac{(1-|\xi_1|^2)d\xi_1}{(1-\overline{\xi_1}z_1)(z_1-\xi_1)} \wedge \frac{z_2 d\overline{\xi_2}}{1-\overline{\xi_2}z_2} = \\
&= \int_{\xi\in\Delta} \frac{(1-|\xi_1|^2)d\xi_1}{(z_1-\xi_1)(1-\overline{\xi_1}z_1)} \log(1-\overline{\xi_2}z_2) \wedge \theta + \\
&\quad +2\pi i \int_{|\xi_2|<|z_1|} \log(1-\overline{\xi_2}z_2) i\theta_{22}(z_1,\xi_2) d\xi_2 d\overline{\xi_2} - \\
&\quad - \int_{|\xi_2|<|\xi_1|} \frac{d\overline{\xi_1}\wedge d\xi_1}{(1-\overline{\xi_1}z_1)^2} \log(1-\overline{\xi_2}z_2) \wedge \theta\,.
\end{aligned} \tag{18}
$$

The third term in the right hand side of (18) is holomorphic and we denote it by

$$
g_1(z_1,z_2) = \int_{|\xi_2|<|\xi_1|} \frac{d\overline{\xi_1}\wedge d\xi_1}{(1-\overline{\xi_1}z_1)^2} \log(1-\overline{\xi_2}z_2) \wedge \theta\,.
$$

Since the other terms in (18) are C^∞ up to the boundary then $g_1(z_1,z_2)$ is C^∞ too. We denote its boundary values by the same function g_1.

The first term in the right hand side of (18) is

$$
\begin{aligned}
&\int_{\xi\in\Delta} \frac{(1-|\xi_1|^2)d\xi_1}{(z_1-\xi_1)(1-\overline{\xi_1}z_1)} \log(1-\overline{\xi_2}z_2) \wedge \theta = \\
&= \int_{\xi\in\Delta} \frac{d\xi_1}{z_1-\xi_1} \log(1-\overline{\xi_2}z_2) \wedge \theta + \int_{\xi\in\Delta} \frac{\overline{\xi_1}d\xi_1}{1-\overline{\xi_1}z_1} \log(1-\overline{\xi_2}z_2) \wedge \theta\,.
\end{aligned} \tag{19}
$$

If we denote by

$$
h_1(z_1,z_2) = -\int_{\xi\in\Delta} \frac{\overline{\xi_1}d\xi_1}{1-\overline{\xi_1}z_1} \log(1-\overline{\xi_2}z_2) \wedge \theta\,,
$$

then $h_1(z_1,z_2)$ is, as g_1 and f_1 holomorphic in D^2 and C^∞ up to the boundary. Putting together (18) and (19) we get for $(z_1,z_2) \in \mathbb{T}^2$

$$
\begin{aligned}
&\int_{|\xi_2|<|\xi_1|} \theta \wedge \frac{d\xi_1}{z_1-\xi_1} \wedge \frac{d\overline{\xi_2}}{\overline{z_2}-\overline{\xi_2}} = \int_{\xi\in\Delta} \frac{d\xi_1}{z_1-\xi_1} \log(1-\overline{\xi_2}z_2) \wedge \theta + \\
&+2\pi i \int_{|\xi_2|<1} \log(1-\overline{\xi_2}z_2) i\theta_{22}(z_1,\xi_2)\, d\xi_2 \wedge d\overline{\xi_2} - f_1(z_1,z_2) - g_1(z_1,z_2) - h_1(z_1,z_2)
\end{aligned} \tag{20}
$$

where $g_1(z_1,z_2)$, $h_1(z_1,z_2)$ and $f_1(z_1,z_2)$ are the boundary values of some $A^\infty(D^2)$ functions.

Analogously if $(z_1, z_2) \in \mathbb{T}^2$

$$\int_{|\xi_1|<|\xi_2|} \theta \wedge \frac{d\xi_1}{z_1 - \xi_1} \wedge \frac{d\overline{\xi_2}}{\overline{z_2} - \overline{\xi_2}} = \int_{\xi \in \Delta} \log(1 - \xi_1 \overline{z_1}) \frac{d\overline{\xi_2}}{\overline{z_2} - \overline{\xi_2}} \wedge \theta + \\ +2\pi i \int_{|\xi_1|<1} \log(1 - \xi_1 \overline{z_1}) i\theta_{11}(\xi_1, z_2) d\xi_1 \wedge d\xi_1 - \\ -f_2(z_1, z_2) - g_2(z_1, z_2) - h_2(z_1, z_2) \tag{21}$$

where g_2, f_2 and h_2 are antiholomorphic and C^∞ up to ∂D^2.

Adding (20) and (21) we have that for any $(z_1, z_2) \in \mathbb{T}^2$

$$\begin{aligned} M(\theta) &= -\frac{1}{2\pi^2} \operatorname{Re} \left\{ i \int_{\xi \in \Delta} d(\log(1 - \xi_1 \overline{z_1}) \log(1 - \overline{\xi_2} z_2) \wedge \theta) \right\} + \\ &\quad + \frac{1}{\pi} \left\{ \int_{|\xi_1|<1} \log|1 - \xi_1 \overline{z_1}| i\theta_{11}(\xi_1, z_2) d\xi_1 \wedge d\overline{\xi_1} + \right. \\ &\quad \left. + \int_{|\xi_2|<1} \log|1 - \overline{\xi_2} z_2| i\theta_{22}(z_1, \xi_2) d\xi_2 \wedge d\overline{\xi_2} \right\} + \\ &\quad + \frac{1}{2\pi^2} \operatorname{Re} \left\{ i(f_1 + g_1 + h_1 + f_2 + g_2 + h_2) \right\}. \end{aligned} \tag{22}$$

The third term in the right hand side of (22) is the boundary values of some smooth pluriharmonic function so

$$\widetilde{M}(\theta) = M(\theta) - \frac{1}{2\pi^2} \operatorname{Re} \left[i(f_1 + g_1 + h_1 + f_2 + g_2 + h_2) \right]$$

is another $C^\infty(\overline{D^2})$ solution of (1), whose boundary values on $\mathbb{T}^2$ are given by

$$\begin{aligned} \widetilde{M}(\theta)(z_1, z_2) &= -\frac{1}{2\pi^2} \operatorname{Re} \left\{ i \int_{\xi \in \Delta} d(\log(1 - \xi_1 \overline{z_1}) \log(1 - \overline{\xi_2} z_2) \wedge \theta) \right\} \\ &\quad + \frac{1}{\pi} \left\{ \int_{|\xi_1|<1} \log|1 - \xi_1 \overline{z_1}| i\theta_{11}(\xi_1, z_2) d\xi_1 \wedge d\overline{\xi_1} + \right. \\ &\quad \left. + \int_{|\xi_2|<1} \log|1 - \overline{\xi_2} z_2| i\theta_{22}(z_1, \xi_2) d\xi_2 \wedge d\overline{\xi_2} \right\}. \blacksquare \end{aligned}$$

2. A Rudin theorem with bounds

In this section we will apply lemma 3 to obtain Rudin's theorem with an explicit bound.

Lemma 4 *Let* $\theta = i\sum_{i,j=1}^{2}\theta_{ij}d\xi_i\wedge d\overline{\xi_j}$ *be a closed positive* $(1-1)$ *current in* D^2 *which satisfies*

(a') there exist $r\in(0,1)$ *such that the support of* θ_{12} *is contained in* $D\times D_r\cup D_r\times D$.

(b')

$$\sup_{\substack{|z_1|>r\\|z_2|>r}}\left\{\int_{|\xi_1|<1}\theta_{11}(\xi_1,z_2)+\int_{|\xi_2|<1}\theta_{22}(z_1,\xi_2)\right\}\leq M$$

(c')

$$A=\int_{D^2}\delta_{\partial D^2}|\theta|<\infty.$$

Then there exist a solution u *to the equation* $i\partial\overline{\partial}u=\theta$ *such that*

(i) $u\leq 0$

(ii) $\|u\|_{L^1(D^2)}\leq C(M+A)$, C *being a universal constant.*

Remark. Note that the bound is in the assertion (ii) of the lemma.

Proof: Lets start by regularizing the current θ by convolution

$$\theta^\varepsilon=(\theta*\chi_\varepsilon)((1-\varepsilon)z)\,,$$

where

$$\chi_\varepsilon(z)=\frac{1}{\varepsilon}\chi\left(\frac{z}{\varepsilon}\right)\,,$$

χ being a positive radial function C^∞ with integral 1 and support contained in the ball $\{|z|<\frac{1}{2}\}$. Let $r<r_1<1$. Let D_{r_1} be the disc of center 0 and radius r_1. Then $\theta^\varepsilon\in C^\infty(\overline{D^2})$, θ^ε is a positive closed $(1-1)$ form such that $\theta^\varepsilon\to\theta$ as $\varepsilon\to 0$ in the sense of currents and

$$\text{Supp }\theta_{12}^\varepsilon\subset D\times D_{r_1}\cup D_{r_1}\times D$$

if ε is small enough.

Lets consider now

$$u^\varepsilon=\widetilde{M}(\theta^\varepsilon)$$

the solution of $i\partial\overline{\partial}u^\varepsilon=\theta^\varepsilon$ defined in lemma 3. $u^\varepsilon\in C^\infty(\overline{D^2})$ and are plurisubharmonic functions ($i\partial\overline{\partial}u^\varepsilon=\theta^\varepsilon\geq 0$). We see now that they are bounded. As they are plurisubharmonic we have to worry only of the values of u at $\mathbb{T}^2$ which are by lemma 3

$$\begin{aligned}u^\varepsilon(z_1,z_2) &= \frac{1}{\pi}\left\{\int_{|\xi_1|<1}\log|1-\xi_1\overline{z_1}|i\theta_{11}^\varepsilon(\xi_1,z_2)d\xi_1\wedge d\overline{\xi_1}+\right.\\ &\quad\left.+\int_{|\xi_2|<1}\log|1-\overline{\xi_2}z_2|i\theta_{22}^\varepsilon(z_1,\xi_2)d\xi_2\wedge d\overline{\xi_2}\right\} \\ &\quad-\frac{1}{2\pi^2}\text{ Re }\left\{i\int_{\xi\in\Delta}d(\log(1-\xi_1\overline{z_1})\log(1-\overline{\xi_2}z_2)\wedge\theta^\varepsilon)\right\}\,.\end{aligned}\tag{23}$$

Because of the support of θ^ε_{12} the third integral in (23) is

$$\int_{\xi\in\Delta} d(\log(1-\xi_1\overline{z_1})\log(1-\overline{\xi_2}z_2)\wedge\theta^\varepsilon) =$$

$$= \int_{\xi\in\Delta\setminus\{(\xi_1,\xi_2)/|\xi_1|=|\xi_2|>r_1\}} d(\log(1-\xi_1\overline{z_1})\log(1-\overline{\xi_2}z_2)\wedge\theta^\varepsilon) =$$

$$= \int_{\substack{|\xi_1|=r_1\\|\xi_2|=r_1}} \log(1-\xi_1\overline{z_1})\log(1-\xi_2 z_2)\wedge\theta^\varepsilon = 0\,.$$

So only the first two terms in the right hand side of (23) can be nonzero. Now

$$u^\varepsilon(z_1,z_2) \le \frac{\log 2}{\pi}\left\{\int_{|\xi_1|<1} i\theta^\varepsilon_{11}(\xi_1,z_2)d\xi_1\wedge d\overline{\xi_1} + \int_{|\xi_2|<1} i\theta^\varepsilon_{22}(z_1,\xi_2)\wedge d\overline{\xi_2}\right\}\,.$$

Thus,

$$u^\varepsilon(z_1,z_2) \le \frac{\log 2}{\pi}\left\{\sup_{|z_2|>r_1}\int_{|\xi_1|\le 1}\theta_{11}(\xi_1,z_2) + \sup_{|z_1|>r_1}\int_{|\xi_2|\le 1}\theta_{11}(z_1,\xi_2)\right\}\,.$$

Then for ε small enough and because of (b')

$$u^\varepsilon(z_1,z_2) \le CM\,.$$

Also if we compute $\|u^\varepsilon\|_{L^1(\mathbb{T}^2)}$ we get that

$$\begin{aligned}\|u^\varepsilon\|_{L^1(\mathbb{T}^2)} \;\le\; & \frac{2}{\pi}\int_{|\xi_1|<1}\int_{z\in\mathbb{T}^2}\Big|\log|1-\xi_1\overline{z_1}|\Big|\theta^\varepsilon_{11}(\xi_1,z_2)d\sigma(z_1)d\sigma(z_2)d\lambda(\xi_1)\\ & +\frac{2}{\pi}\int_{|\xi_2|<1}\int_{z\in\mathbb{T}^2}\Big|\log|1-\overline{\xi_2}z_2|\Big|\theta^\varepsilon_{22}(z_1,\xi_2)d\sigma(z_1)d\sigma(z_2)d\lambda(\xi_2)\end{aligned}$$

where $d\sigma$ is the Haar measure in $\mathbb{T}$ and $d\lambda$ the Lebesgue measure on D. Then, for ε small enough, $\|u^\varepsilon\|_{L^1(\mathbb{T}^2)} \le C_1 M$.

We estimate now $\|u^\varepsilon\|_{L^1(\partial D^2)}$. Let $(z_1,z_2)\in D\times\mathbb{T}$, by the Poisson-Jensen formula

$$u^\varepsilon(z_1,z_2) = \int_{\xi\in D} G(\xi_1,z_1)\Delta_1 u^\varepsilon(\xi_1,z_2) + \int_{\xi\in\mathbb{T}} P(\xi_1,z_1)u^\varepsilon(\xi_1,z_2)$$

where G is the Green function in the disc and P the Poisson kernel, both are integrable, so

$$\int_{D\times\mathbb{T}} |u^\varepsilon| \le C_2\int_{D\times\mathbb{T}}\theta^\varepsilon_{11} + C_2\|u^\varepsilon\|_{L^1_{\mathbb{T}^2}}$$

So finally,

$$\|u^\varepsilon\|_{L^1(\partial D^2)} \le C_3 M \tag{24}$$

Lets compute now the $L^1(D^2)$ norm of u^ε. It is known that whenever $z=(z_1,z_2)\in D^2$

$$u^\varepsilon(z) = \int_{\xi\in D^2} G(\xi,z)\Delta u^\varepsilon(\xi) + \int_{\xi\in\partial D^2} P(\xi,z)u^\varepsilon(\xi)$$

where G is the Green function in the bidisc and P the Poisson kernel. It is easily seen that

$$\int_{z\in D^2} |G(\xi,z)|\,d\lambda(z) \le K\delta_{\partial D^2}(\xi) \tag{25}$$

and

$$\int_{z\in D^2} P(\xi,z)\,d\lambda(z) \le K.$$

Let us check (25) for instance. Consider the function

$$f(z) = -\frac{(1-|z_1|^2)(1-|z_2|^2)}{(1-|z_2|^2)+(1-|z_1|^2)}.$$

This function belongs to $C^2(D^2)\cap C(\overline{D^2})$. It vanishes on the boundary and moreover

$$\Delta f = 4\frac{(1-|z_1|^2)^2+(1-|z_2|^2)^2}{((1-|z_2|^2)+(1-|z_1|^2))^2} \ge 1.$$

Thus,

$$\int_{z\in D^2} |G(\xi,z)|\,d\lambda(z) \le -\int_{z\in D^2} G(\xi,z)\Delta f(z)\,d\lambda(z) = -f(\xi) \le K\delta_{\partial D^2}(\xi).$$

So finally

$$\int_{z\in D^2} |u^\varepsilon| \le K\int_{\xi\in D^2} \delta_{\partial D^2}(\xi)|\theta^\varepsilon(\xi)| + K\|u^\varepsilon\|_{L^1(\partial D^2)}.$$

Now, because of hypothesis (c') and (24) we get

$$\|u^\varepsilon\|_{L^1(D^2)} \le C(M+A),$$

C being a universal constant. So there is a sequence $\varepsilon_n \to 0$, such that $u^{\varepsilon_n} \to v$ in the sense of measures, v being a bounded measure. Thus it converges also in the sense of currents and $i\partial\overline{\partial}u^{\varepsilon_n} \to i\partial\overline{\partial}v$. So $i\partial\overline{\partial}v = \theta$ and v is a plurisubharmonic function such that $\|v\|_{L^1(D^2)} \le C(M+A)$. The inequality $v \le M$ follows from the submean-value property of v. Taking $u = v - M$ we obtain the desired plurisubharmonic function. ∎

3. End of the proof of theorem 1'

If θ is a current wich satisfies the hypothesis of theorem 1' then $\theta^n = \theta(r_n^1\xi_1, r_n^2\xi_2)$ is a dilated current such that $\theta^n \to \theta$ in the sense of currents when $n \to \infty$. Moreover hypothesis (a) (b) and (c) on θ imply that θ^n satisfies (a') (b') and (c') of lemma 4. So we have a solution u^n of $\partial\overline{\partial}u^n = \theta^n$ such that

(i) $u^n \le 0$

(ii) $\|u^n\|_{L^1(D^2)} \le C(M+A)$. C being a universal constant.

As a consequence, with the same argument as in the end of the proof of lemma 4, choosing a subsequence $n_k \to \infty$, we get a solution $u = \lim_{n_k \to \infty} u^{n_k}$ of (1), such that u is negative. ∎

III. Proof of proposition 2 and theorem 3

1. Proof of proposition 2

Let X be a divisor in D^n, then $X = (m_k, X_k)$ where X_k are the irreducible components (i.e. the connected components of the regular points of X) and m_k the multipliciy of each X_k. Consider now $X' = (m_k, A_k)$ the irreducible components of X such that they cut $\vartheta(\mathbb{T}^n)$ and $X'' = (m_k, B_k)$ the irreducible components of X that do not cut $\vartheta(\mathbb{T}^n)$. By Rudin's theorem, there is a function f_2 such that $Z(f_2) = X''$ and $f_2 \in H^\infty(D^n)$. Now consider h the bounded holomorphic function given, $Z(h)$ is a divisor $Y = (n_k, Y_k)$, we separate again the irreducible components $Y' = (n_k, C_k)$ that cut $\vartheta(\mathbb{T}^n)$ and $Y'' = (n_k, D_k)$ the components that do not cut. Take one component A_k of X', there is one component C_k of Y', such that they coincide on $\vartheta(\mathbb{T}^n)$. As this is an open set, they coincide along the whole D^n. So $X' = Y'$, but Rudin's theorem states that there is an holomorphic bounded function $h_2 \in H^\infty(D^n)$ such that $Z(h_2) = Y''$ and moreover $1/h_2$ is bounded in a neighbourhood of $\mathbb{T}^n$, so finally

$$f = f_2 h / h_2$$

is a bounded holomorphic function such that $Z(f) = X$. ∎

2. Proof of theorem 3

Let X be a divisor in D^2, as in the proof of proposition 2, we consider $X' = (m_k, A_k)$ the irreducible components that do intersect $\Delta \cap \vartheta(\mathbb{T}^2)$ and $X'' = (m_k, C_k)$ the components that do not intersect. By theorem 1, there is a function $f_2 \in H^\infty(D^2)$ such that $Z(f_2) = X''$. Now for any irreducible component $A_k \in X'$ the intersection with Δ must be a curve γ_k. It can not be a point because $\{1 > |z_1| > |z_2|\}$ and $\{1 > |z_2| > |z_1|\}$ are pseudoconvex domains. There is an irreducible component Y_k of $Y = Z(h)$, $h \in H^\infty(D^2)$, such that the intersection of Y_k with Δ is γ_k by the hypothesis of the theorem. But γ_k is a determinant set in Y_k, so $A_k = Y_k$. Thus if we consider

$$f = f_2 h$$

then $X \subset Z(f)$. ∎

References

[And] Andersson, M.: *Solution formulas for the $\partial\overline{\partial}$-equation and weighted Nevanlinna classes in the polydisc.* Bull. Sc. Math. **109**, 135–154 (1985).

[Be] Berndtsson, B.: *Integral formulas for the $\partial\overline{\partial}$-equation and zeros of bounded holomorphic fucntions in the unit ball.* Math. Ann. **249**, 163–176 (1980).

[Ch1] Charpentier, P.: *Sur la formule de Jensen et les zeros des fonctions holomorphes dans le polydisque.* Math. Ann. **242**, 27–46 (1979).

[Ch2] Charpentier, P.: *Caractérisations des zeros des fonctions de certaines classes de bidisque.* Ann. Inst. Fourier, Grenoble **34**, 57–98 (1984).

[Ch3] Charpentier, P.: *Sur les zeros des fonctions de type Nevanlinna dans le bidisque.* Lecture Notes in Math. **1094**, 32–43 (1984).

[Che] Chee, P. S.: *Zero sets and extensions of bounded holomorphic functions in polydiscs.* Proc. Amer. Math. Soc. **60**, 109–115 (1976).

[Ga] Garnett, J.B.: *"Bounded analytic functions".* Academic Press. San Diego, California.

[Le] Lelong, P.: *"Fonctionelles analytiques et fonctions entières (n variables)".* Montreal. Les presses de l'université de Montreal, 1968.

[Ru1] Rudin, W.: *Zero-sets in polydiscs.* Bull. Amer. Math. Soc. **73**, 580–583 (1967).

[Ru2] Rudin, W.: *"Function theory in the unit ball of $\mathbb{C}^n$".* Springer-Verlag. New York.

[Za] Zarantonello, S.E.: *The multiplicative Cousin problem and a zero set for the Nevanlinna class in the polydisc.* Trans. Amer. Math. Soc. **200**, 291-313 (1974).

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